Frequency Downshift in a Viscous Fluid

John D. Carter

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Major Collaborators

- Isabelle Butterfield (Seattle University)
- Alex Govan (Seattle University)
- Diane Henderson (Penn State University)
- Harvey Segur (University of Colorado at Boulder)

Waves I'd Like to Model

Photo from Shawn at Videezy.com.

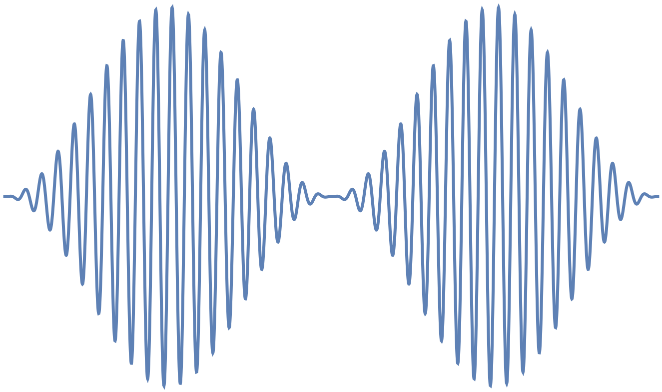
Modeling waves like those is too difficult for me because of:

- Wave breaking
- Air trapped in the fluid
- Vorticity
- Wind
- Interactions with the seafloor
- ...

Waves I'm Going to Talk About Today

Photo from http://teachersinstitute.yale.edu/curriculum/units/2008/5/08.05.06.x.html.

Waves I'm Going to Talk About Today

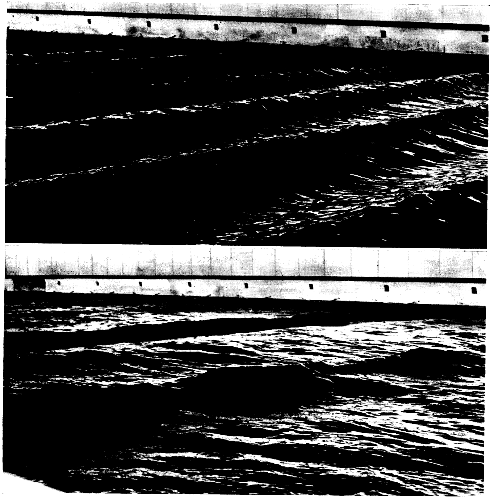


Two-dimensional modulated wave trains.

Select Background

Benjamin & Feir (1967) Theory and Experiments

Benjamin-Feir Instability



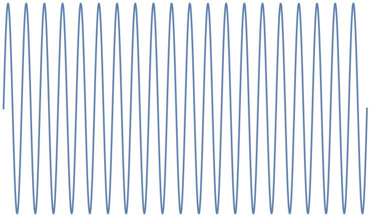
(a)

(b)

FIGURE 1. Photographs of a progressive wavetrain at two stations, illustrating disintegration due to instability: (a) view near to wavemaker; (b) view at 200 ft. farther from wavemaker. Fundamental wavelength, 7·2 ft.

Benjamin-Feir Instability

A time series that initially has the form



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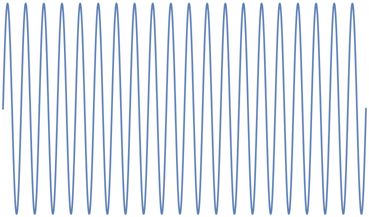
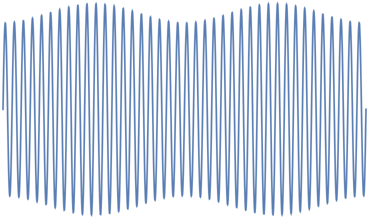


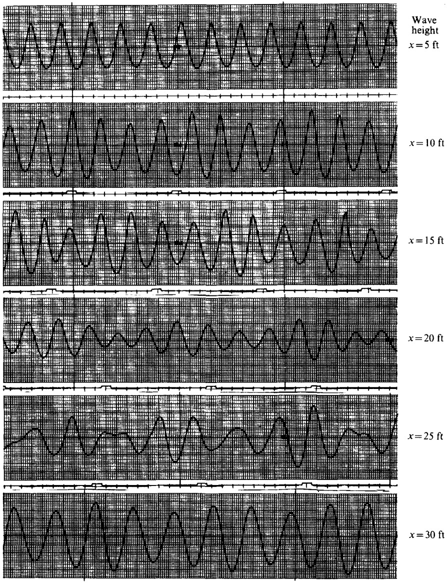
will evolve into



due to the Benjamin-Feir (or modulational) instability.

Yuen, Lake, Rungaldier, & Ferguson (1977) Experiments

Frequency Downshifting



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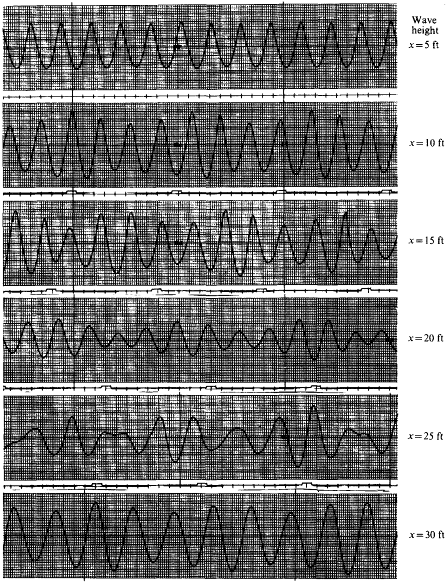


13 peaks

13 peaks

13 peaks

13 peaks

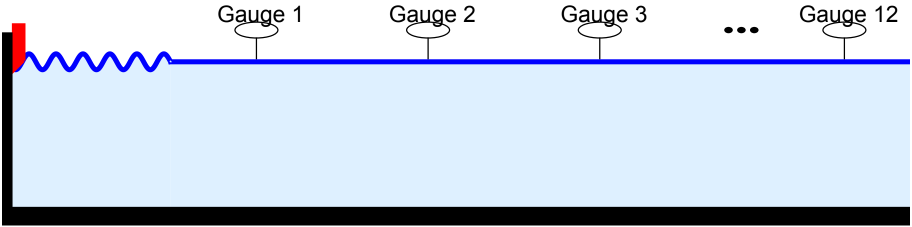
?? peaks

10 peaks

Segur *et al.* (2005) Theory and Experiments

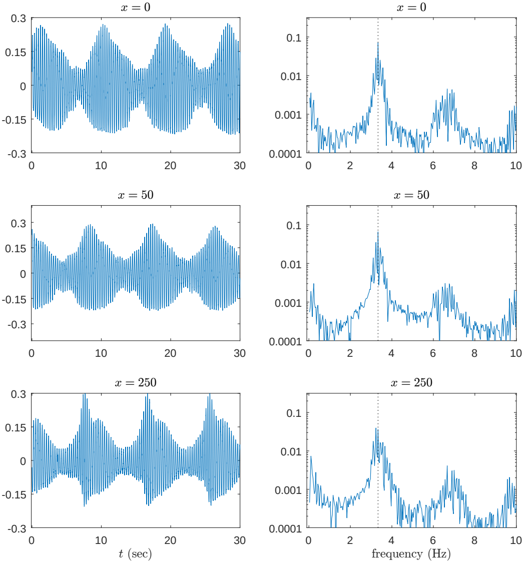
Basic Experimental Setup

Figure not to scale!



Experiments conducted by Diane Henderson (Penn State University).

Experimental Measurements



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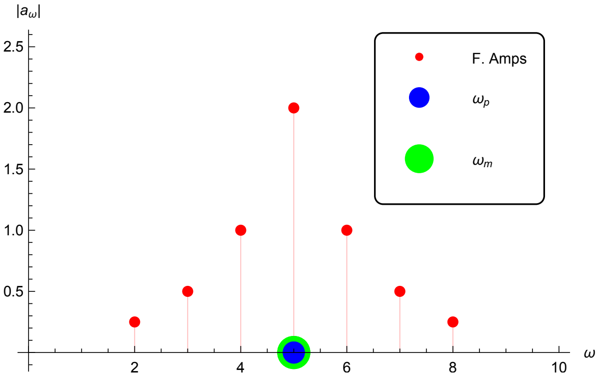
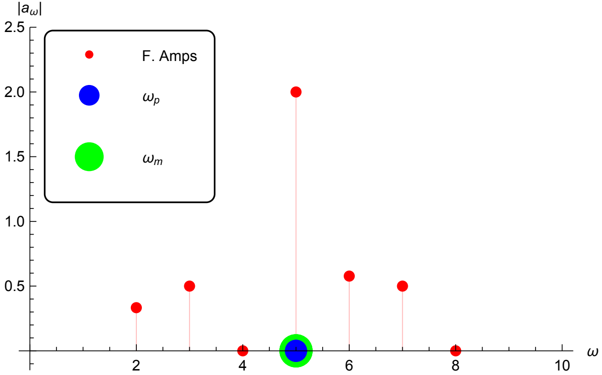
A wave train is said to exhibit frequency downshifting if ω_m or ω_p decreases monotonically as it travels down the tank.

Frequency Downshift



Frequency downshift in both the spectral peak and spectral mean senses.

Frequency Downshift



Frequency downshift in the spectral peak sense.

More Experimental Background

Segur *et al.* (2005) showed

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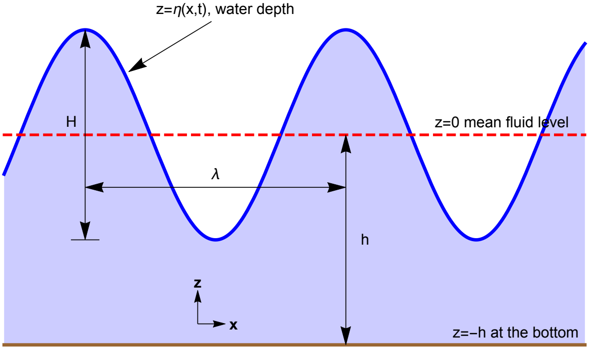
Our goal is to provide a mathematical justification for these observations without relying on wind or wave breaking effects.

Theoretical Background

Physical System



- $\eta = \eta(x, t)$ represents the surface displacement
- $\phi = \phi(x, z, t)$ represents the velocity potential

Governing Equations

The equations for a two-dimensional, infinitely deep, inviscid, irrotational, incompressible fluid are

$$\phi_{xx} + \phi_{zz} = 0, \quad \text{for} \quad -\infty < z < \eta(x,t)$$

$$\eta_t + \phi_x \eta_x - \phi_z = 0, \quad \text{for} \quad z = \eta(x,t)$$

$$\phi_t + g\eta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) = 0, \quad \text{for} \quad z = \eta(x,t)$$

$$\phi_z \to 0, \quad \text{as} \quad z \to -\infty$$

Approximate Models

In 1966, Zakharov assumed

$$\eta(x,t)=\epsilon B e^{ik_0 x-i\omega_0 t}+\epsilon^2 B_2 e^{2(ik_0 x-i\omega_0 t)}+\epsilon^3 B_3 e^{3(ik_0 x-i\omega_0 t)}+\cdots+c.c.$$

$$\phi(x,z,t)=\epsilon A_1 e^{k_0 z+ik_0 x-i\omega_0 t}+\epsilon^2 A_2 e^{2(k_0 z+ik_0 x-i\omega_0 t)}+\epsilon^3 A_3 e^{3(k_0 z+ik_0 x-i\omega_0 t)}+\cdots+c.c.$$

in order to study the evolution of modulated wave trains.

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in order to study the evolution of modulated wave trains. Here

- $\epsilon = 2|a_0|k_0 \ll 1$ is the dimensionless wave steepness
- a_0 represents a typical amplitude
- k_0 represents the wave number of the carrier wave
- ω_0 represents the frequency of the carrier wave
- The A's depend on $X = \epsilon x$, $Z = \epsilon z$, and $T = \epsilon t$
- The B's depend on X and T
- $c.c.$ stands for complex conjugate

NLS Equation

This led to the nonlinear Schrödinger (NLS) equation

$$2i\omega_0\left(B_T + \frac{g}{2\omega_0}B_X\right) + \epsilon\left(\frac{g}{4k_0}B_{XX} + 4gk_0^3|B|^2B\right) = 0$$

where

$$\omega_0^2 = gk_0$$

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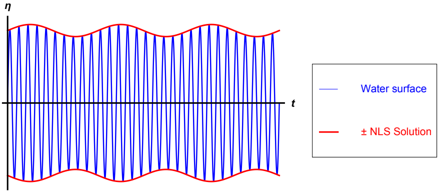
B models the evolution of the red curves (the "envelope").



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Properties

- NLS preserves mass, $\mathcal{M}$
- NLS preserves linear momentum, $\mathcal{P}$
- NLS preserves the spectral mean, ω_m

Dysthe System

In 1979, Dysthe generalized Zakharov's work by assuming

$$\eta(x,t)=\epsilon^3\bar{\eta}+\epsilon B e^{ik_0x-i\omega_0t}+\epsilon^2 B_2 e^{2(ik_0x-i\omega_0t)}+\epsilon^3 B_3 e^{3(ik_0x-i\omega_0t)}+\cdots+c.c.$$

$$\phi(x,z,t)=\epsilon^2\bar{\phi}+\epsilon A_1 e^{k_0z+ik_0x-i\omega_0t}+\epsilon^2 A_2 e^{2(k_0z+ik_0x-i\omega_0t)}+\epsilon^3 A_3 e^{3(k_0z+ik_0x-i\omega_0t)}+\cdots+c.c.$$

Dysthe System

This led to what is now known as the Dysthe system

$$2i\omega_0\left(B_T+\frac{g}{2\omega_0}B_X\right)+\epsilon\left(\frac{g}{4k_0}B_{XX}+4gk_0^3|B|^2B\right)$$

$$+\epsilon^2\left(-i\frac{g}{8k_0^2}B_{XXX}+2igk_0^2B^2B_X^*+12igk_0^2|B|^2B_X+2k_0\omega_0B\bar{\phi}_{0X}\right)=0,\ \text{at } Z=0$$

$$\bar{\phi}_{0Z}=2\omega_0\left(|B|^2\right)_X, \qquad \text{at } Z=0$$

$$\bar{\phi}_{0XX}+\bar{\phi}_{0ZZ}=0, \qquad \text{for } -\infty<Z<0$$

$$\bar{\phi}_{0Z}\to0, \qquad \text{as } Z\to-\infty$$

Dysthe System

Properties

- The Dysthe system preserves $\mathcal{M}$
- The Dysthe system does not preserve $\mathcal{P}$
- The Dysthe system does not preserve ω_m

Derivation of the Viscous Dysthe System

Governing Equations with Weak Viscosity

Dias *et al.* (2008) derived a weakly viscous generalization of the Euler equations

$$\phi_{xx} + \phi_{zz} = 0, \quad \text{for} \quad -\infty < z < \eta(x,t)$$

$$\eta_t + \phi_x \eta_x - \phi_z = 2\bar{\nu}\eta_{xx}, \quad \text{for} \quad z = \eta(x,t)$$

$$\phi_t + g\eta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) = -2\bar{\nu}\phi_{zz}, \quad \text{for} \quad z = \eta(x,t)$$

$$\phi_z \to 0, \quad \text{as} \quad z \to -\infty$$

Where $\bar{\nu}$ is the kinematic viscosity.

Governing Equations with Weak Viscosity

Wu *et al.* (2006) studied the following *ad-hoc* dissipative generalization of the Euler equations

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$$\eta_t + \phi_x \eta_x - \phi_z = 0, \quad \text{for} \quad z = \eta(x,t)$$

$$\phi_t + g\eta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) = -\beta\phi_{zz}, \quad \text{for} \quad z = \eta(x,t)$$

$$\phi_z \to 0, \quad \text{as} \quad z \to -\infty$$

Where β is the coefficient of dissipation.

Solution Ansatz

Generalizing the work of Dysthe, assume

$$\eta(x,t)=\epsilon^3\bar{\eta}+\epsilon B e^{i\omega_0 t-ik_0x}+\epsilon^2 B_2 e^{2(i\omega_0 t-ik_0x)}+\epsilon^3 B_3 e^{3(i\omega_0 t-ik_0x)}+\cdots+c.c.$$

$$\phi(x,z,t)=\epsilon^2\bar{\phi}+\epsilon A_1 e^{k_0z+i\omega_0 t-ik_0x}+\epsilon^2 A_2 e^{2(k_0z+i\omega_0 t-ik_0x)}+\epsilon^3 A_3 e^{3(k_0z+i\omega_0 t-ik_0x)}+\cdots+c.c.$$

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Here

- $\epsilon = 2|a_0|k_0 \ll 1$ is the dimensionless wave steepness
- a_0 represents a typical amplitude
- $\omega_0 > 0$ represents the frequency of the carrier wave
- $k_0 > 0$ represents the wave number of the carrier wave
- The A_j's and $\bar{\phi}$ depend on $X = \epsilon x$, $Z = \epsilon z$, $T = \epsilon t$
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- $\bar{\nu} = \epsilon^2\nu$

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Dissipative NLS Equation

At $\mathcal{O}(\epsilon^3)$, this leads to the dissipative NLS (dNLS) equation (studied by Segur *et al.*, (2005), derived by Dias *et al.*, (2008)).

$$2i\omega_0\big(B_T + \frac{g}{2\omega_0}B_X\big) + \epsilon\Big(-\frac{g}{4k_0}B_{XX} - 4gk_0^3|B|^2B + 4ik_0^2\omega_0\nu B\Big) = 0$$

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Properties

- dNLS does not preserve $\mathcal{M}$
- dNLS does not preserve $\mathcal{P}$
- dNLS preserves ω_m

Viscous Dysthe System

At $\mathcal{O}(\epsilon^4)$, this leads to the viscous Dysthe (vDysthe) system

$$2i\omega_0\left(B_T+\frac{g}{2\omega_0}B_X\right)+\epsilon\left(\frac{g}{4k_0}B_{XX}+4gk_0^3|B|^2B+4ik_0^2\omega_0\nu B\right)$$

$$+\epsilon^2\left(-i\frac{g}{8k_0^2}B_{XXX}+2igk_0^2B^2B_X^*+12igk_0^2|B|^2B_X+2k_0\omega_0B\bar{\phi}_{0X}-8k_0\omega_0\nu B_X\right)=0,\text{ at } Z=0$$

$$\bar{\phi}_{0Z}=2\omega_0\left(|B|^2\right)_X, \qquad \text{at } Z=0$$

$$\bar{\phi}_{0XX}+\bar{\phi}_{0ZZ}=0, \qquad \text{for } -\infty<Z<0$$

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Change Variables

$$k_0 B(X,T) = \tilde{B}(\xi,\chi)$$

$$\frac{k_0^2}{\omega_0} A(X,Z,T) = \tilde{A}(\xi,\chi,\zeta)$$

$$\frac{k_0^2}{4\omega_0} \bar{\phi}_0(X,Z,T) = \tilde{\Phi}(\xi,\chi,\zeta)$$

$$\frac{4k_0^2}{\omega_0} \nu = \delta$$

$$\chi = \epsilon k_0 X$$

$$\xi = \omega_0 T - 2k_0 X$$

$$\zeta = k_0 Z$$

The Dimensionless Viscous Dysthe System

$$iB_\chi + B_{\xi\xi} + 4|B|^2 B + i\delta B + \epsilon\Big(-8iB^2 B_\xi^* - 32i|B|^2 B_\xi - 16B\Phi_\xi + 5\delta B_\xi\Big) = 0, \qquad \text{at } \zeta = 0$$

$$\Phi_\zeta = -\Big(|B|^2\Big)_\xi, \qquad \text{at } \zeta = 0$$

$$4\Phi_{\xi\xi} + \Phi_{\zeta\zeta} = 0, \qquad \text{for} -\infty < \zeta < 0$$

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There is only one free parameter, δ, in this system.

Properties of the Viscous Dysthe System

The vDysthe system does not preserve $\mathcal{M}$ nor $\mathcal{P}$.

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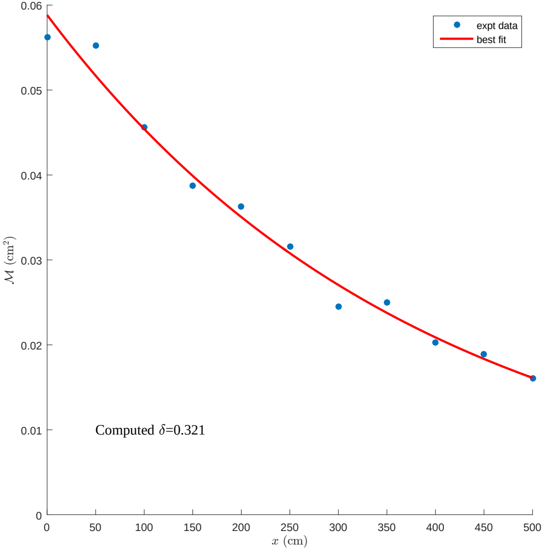
The χ dependency of $\mathcal{M}$ is given by

$$\mathcal{M}_\chi = -2\delta\mathcal{M} - 10\frac{\delta}{\omega_0}\mathcal{P}$$

At leading order in ϵ, this relationship determines δ.

Determining δ



Properties of the Viscous Dysthe System

The viscous Dysthe system does not preserve the spectral mean

$$\left(\omega_m\right)_\chi = \left(\frac{\mathcal{P}}{\mathcal{M}}\right)_\chi = -\frac{10\delta}{\omega_0 \mathcal{M}^2}\Big(\mathcal{M}\mathcal{Q} - \mathcal{P}^2\Big) - \frac{16}{\omega_0}\frac{\mathcal{R}}{\mathcal{M}}$$

where

$$\mathcal{Q} = \frac{\epsilon^4 \omega_0^2}{k_0^2}\frac{1}{\epsilon\omega_0 L}\int_0^{\epsilon\omega_0 L} |B_\xi|^2 d\xi$$

$$\mathcal{R} = \frac{\epsilon^4 \omega_0^2}{k_0^2}\frac{1}{\epsilon\omega_0 L}\,\mathrm{Im}\left(\int_0^{\epsilon\omega_0 L} |B|^2 B^* B_{\xi\xi} d\xi\right)$$

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where

$$\mathcal{Q} = \frac{\epsilon^4\omega_0^2}{k_0^2}\frac{1}{\epsilon\omega_0 L}\int_0^{\epsilon\omega_0 L}|B_\xi|^2 d\xi$$

$$\mathcal{R} = \frac{\epsilon^4\omega_0^2}{k_0^2}\frac{1}{\epsilon\omega_0 L}\,\mathrm{Im}\left(\int_0^{\epsilon\omega_0 L}|B|^2 B^* B_{\xi\xi} d\xi\right)$$

The Cauchy-Schwarz inequality establishes that
$(\mathcal{M}\mathcal{Q} - \mathcal{P}^2) \geq 0$.

Plane-Wave Solutions of the Viscous Dysthe System

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The viscous Dysthe system admits plane-wave solutions given by

$$B(\xi,\chi) = B_0 \exp\left(w_r(\chi) + i w_i(\chi)\right)$$

$$\Phi(\xi,\chi) = 0$$

where

$$w_r(\chi) = -\delta\chi$$

$$w_i(\chi) = \frac{2B_0^2}{\delta}\left(e^{-2\delta\chi} - 1\right)$$

and B_0 is a real parameter.

Stability of Plane-Wave Solutions

Consider perturbed solutions of the form

$$B_{\text{pert}}(\xi,\chi) = \Big(B_0 + \mu u(\xi,\chi) + i\mu v(\xi,\chi) + \mathcal{O}(\mu^2)\Big) \exp\Big(w_r(\chi) + i w_i(\chi)\Big)$$

$$\Phi_{\text{pert}}(\xi,\chi,\zeta) = 0 + \mu p(\xi,\chi,\zeta) + \mathcal{O}(\mu^2)$$

where

- μ is a small real parameter
- u, v, and p are real-valued functions

Plane-Wave Stability **Observations**

The non-transient linear stability problem gives (in physical coordinates)

$$\eta(x,t) = d_0 \exp\left(i\omega_0 t + if_0(x) - 4\bar{\nu}\frac{k_0^3}{\omega_0}x\right)$$
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where d_j are complex constants and f_j are real-valued functions.

Plane-Wave Stability **Observations**

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- This suggests that FD will be observed in the higher harmonics before it is observed in the fundamental.

Comparisons with Experiments

No FD Experiment Fourier Amplitudes

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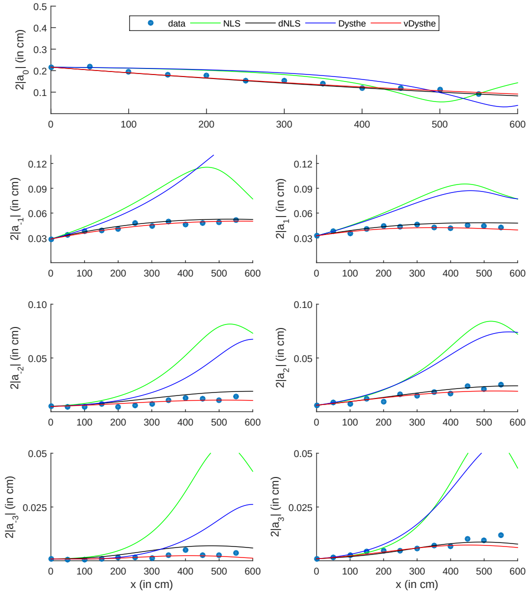


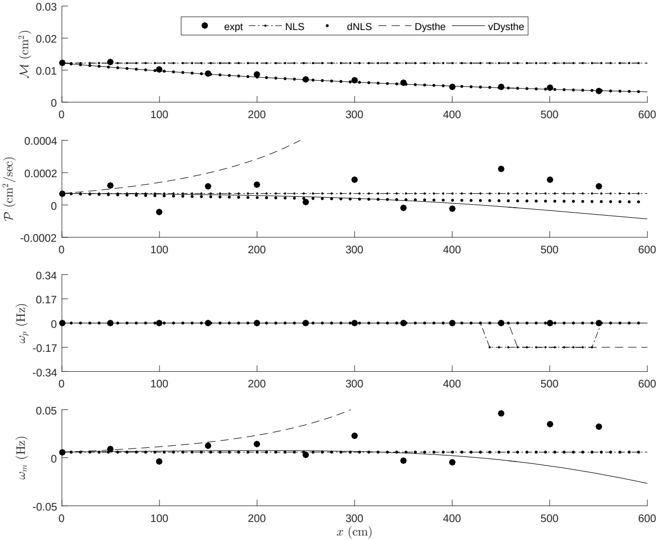
No FD Experiment Fourier Amplitudes

We quantitatively measure the differences between experimental data and PDE predictions via

$$\text{diff}_n = \sum_{j=1}^{11} \Big| 2\big|a_n^{\text{expt}}(50j)\big| - 2\big|a_n^{\text{PDE}}(50j)\big| \Big|$$

PDE	$n=0$	$n=-1$	$n=1$	$n=-2$	$n=2$	$n=-3$	$n=3$
NLS	0.536	0.782	0.652	0.621	0.585	0.393	0.423
Dysthe	0.158	0.055	0.065	0.085	0.040	0.051	0.022
dNLS	0.617	0.853	0.565	0.358	0.494	0.124	0.369
vDysthe	0.136	0.036	0.050	0.037	0.050	0.015	0.028

No FD Experiment Quantities



FD 1 Experiment Fourier Amplitudes

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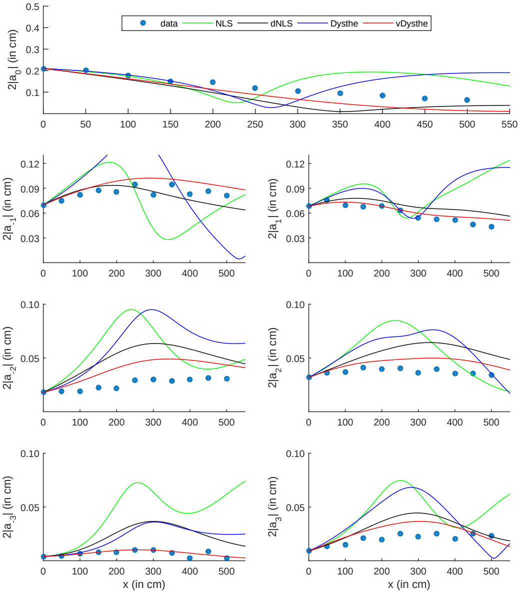

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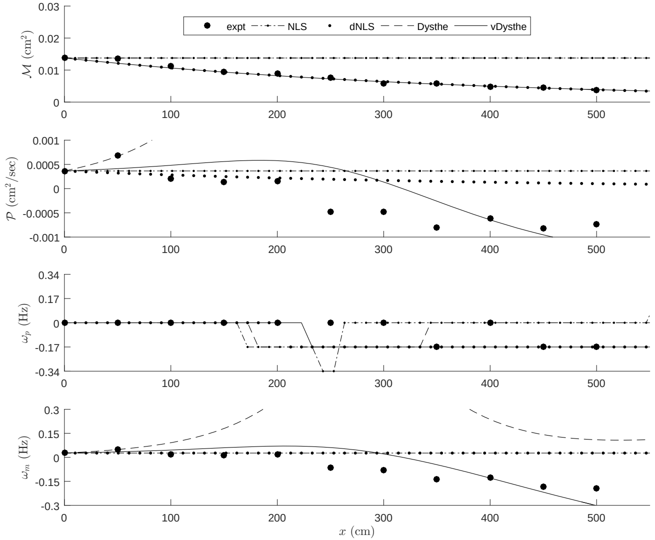
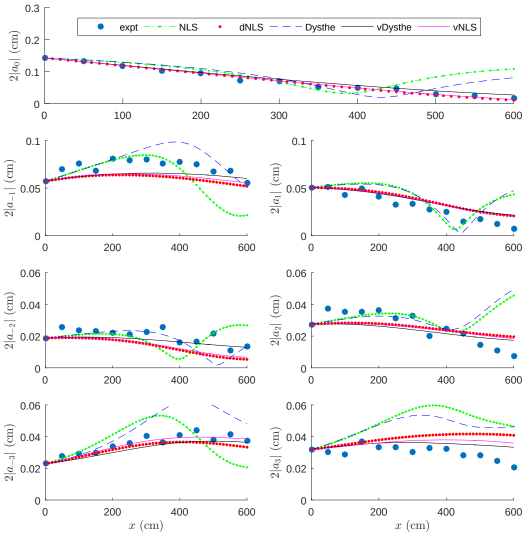
PDE	$n=0$	$n=-1$	$n=1$	$n=-2$	$n=2$	$n=-3$	$n=3$
NLS	0.583	0.293	0.298	0.309	0.215	0.245	0.395
Dysthe	0.480	0.092	0.095	0.244	0.175	0.162	0.106
dNLS	0.554	0.406	0.297	0.380	0.225	0.159	0.227
vDysthe	0.396	0.094	0.041	0.146	0.082	0.013	0.083

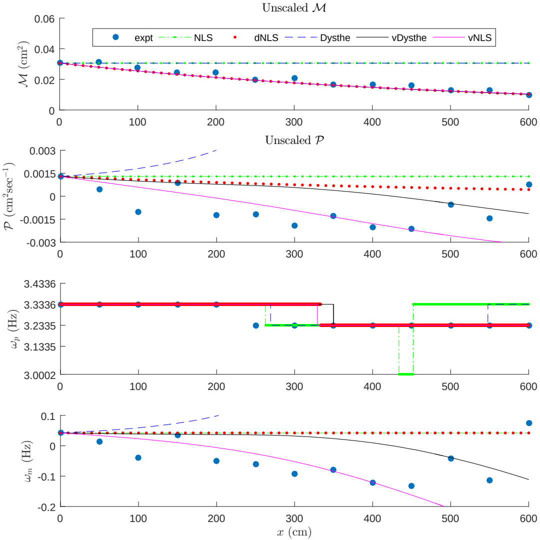
FD 1 Experiment Quantities



FD 2 Experiment Fourier Amplitudes



FD 2 Experiment Quantities



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$$iB_\chi + B_{\xi\xi} + 4|B|^2 B + \epsilon c_1 B\left(|B|^2\right)_\xi = 0$$

where c_1 is a real constant.

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The (ad-hoc) dimensionless Schober & Islas equation is given by

$$iB_\chi + B_{\xi\xi} + 4|B|^2B + i\delta B + \epsilon\left(-8iB^2B_\xi^* - 32i|B|^2B_\xi - 16B\Phi_\xi - ic_2B\Phi_\xi\right) = 0$$

where c_2 is a real constant.

Comparisons with Other FD Theories

Using the norm

$$\text{error} = \sum_{n=-11}^{11} \sum_{j=1}^{10} \left| \left| a_n^{\text{expt}}(50j) \right| - \left| a_n^{\text{PDE}}(50j) \right| \right|$$

we find

PDE	error
NLS	0.1016
Dysthe	0.0893
dNLS	0.0492
vDysthe	0.0459
(optimal) Schober	0.0479
(optimal) Gordon	0.0769

Current Work

- Conducting additional experiments to test the robustness of the viscous Dysthe system
  - Paper on arXiv.org by Kimmoun *et al.* (2017)
- Generalizing the theory of Gramstad & Trulsen (2011)
- Adding in full dispersion/viscosity
- Generalizing the work of Dias *et al.* (2008)